MATH 5061 Solution to Problem Set 3[1]

1. Suppose that (M^n, g) is a connected Riemannian manifold with $n \geq 3$ such that there exists a function $f : M \to \mathbb{R}$ such that $K(\sigma) = f(p)$ for all two-dimensional subspace $\sigma \subset T_pM$. Show that f must be a constant function on M.(*Hint: use the second Bianchi identity*)

   Solution:

   Recall the corollary in the lecture. It says sectional curvature $K(\sigma) \equiv c$ for all $\sigma \in T_pM$ if and only if $R(X, Y, Z, W) = c(\langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle)$. So we have

$$R_p(X, Y, Z, W) = f(p)(\langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle)$$

   Again, we can work at normal coordinate. Let $e_1, \cdots, e_n$ to be the normal coordinate at p. Use the properties $\frac{\partial}{\partial x^i}\langle e_j, e_k\rangle = 0$ at p for any $1 \leq i, j, k \leq n$,

   we have

$$\begin{aligned}(\nabla_{e_i} R)(e_j, e_k, e_l, e_m) &= \frac{\partial}{\partial x^i}\left(f(p)(\langle e_j, e_l\rangle \langle e_k, e_m\rangle - \langle e_j, e_m\rangle \langle e_k, e_l\rangle)\right) \\ &= \frac{\partial f}{\partial x^i}(p)(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\end{aligned}$$

   Now since $n \geq 3$, for any i, we can choose j, k such that i, j, k are all different with each other. Choose $l = j, m = k$ and use second Bianchi Identity, we have

$$0 = \frac{\partial f}{\partial x^i}(1-0) + \frac{\partial f}{\partial x^j}(0-0) + \frac{\partial f}{\partial x^k}(0-0) = \frac{\partial f}{\partial x^i}$$

   So $\nabla f = 0$ at p. Since p is arbitrary, we know f is a constant function.

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2. A Riemannian manifold (M^n, g) is called *Einstein manifold* if there exists a smooth function $\lambda : M \to \mathbb{R}$ such that $\mathrm{Ric}(X, Y) = \lambda\langle X, Y\rangle$ for any vector fields $X, Y \in \Gamma(TM)$.

   (a) Suppose (M^n, g) is a connected Einstein manifold with $n \geq 3$, show that λ must be a constant function.

   (b) Suppose (M^3, g) is a connected 3-dimensional Einstein manifold. Show that M has constant sectional curvature.

Solution:

(a) Let $(x_1, \cdots, x_n)$ to be a local coordinate with $e_i = \frac{\partial}{\partial x_i}$. We write $R(e_i, e_j, e_k, e_l) = R_{ijkl}$ and $\mathrm{Ric}(e_i, e_j) = \mathrm{Ric}_{ij}$ for short. So our condition says $\mathrm{Ric}_{ij} = \lambda g_{ij}$. The second Bianchi identity can be written as

$$\nabla_{e_i} R_{jklm} + \nabla_{e_j} R_{kilm} + \nabla_{e_k} R_{ijlm} = 0$$

We multiply g^{jl}, g^{km} to the both side of the above identity and take sum over j, l, k, m, and using coderivative of metric is 0, we have

$$\begin{aligned}
0 &= g^{jl}g^{km}\nabla_{e_i} R_{jklm} + g^{jl}g^{km}\nabla_{e_j} R_{kilm} + g^{jl}g^{km}\nabla_{e_k} R_{ijlm} \\
&= \nabla_{e_i}\left(g^{jl}g^{km}R_{jklm}\right) + \nabla_{e_j}\left(g^{jl}g^{km}R_{kilm}\right) + \nabla_{e_k}\left(g^{jl}g^{km}R_{ijlm}\right) \\
&= \nabla_{e_i}\left(g^{jl}\mathrm{Ric}_{jl}\right) + \nabla_{e_j}\left(-g^{jl}\mathrm{Ric}_{il}\right) + \nabla_{e_k}\left(-g^{km}\mathrm{Ric}_{im}\right) \quad \text{(Definition of Ric)} \\
&= \nabla_{e_i}\left(g^{jl}g_{jl}\lambda\right) - \nabla_{e_j}\left(g^{jl}g_{il}\lambda\right) - \nabla_{e_k}\left(g^{km}g_{im}\lambda\right) \quad (\mathrm{Ric}_{ij} = \lambda g_{ij}) \\
&= \nabla_{e_i}(n\lambda) - \nabla_{e_j}(\delta_i^j\lambda) - \nabla_{e_k}(\delta_i^k\lambda) = (n-2)\frac{\partial}{\partial x_i}\lambda
\end{aligned}$$

where we've used Einstein summation convention.

Hence $\nabla\lambda \equiv 0$ on M. So λ is a constant function since M is connected.

(b) Let e_1, e_2 be any orthogonal vectors at p. So the section curvature at the plane spanned by e_1, e_2 is R_{1212}. Let's choose e_3 to form a orthonormal basis of T_pM with e_1, e_2 and extend them to a local frame. Note that

$$\begin{aligned}
&\mathrm{Ric}_{11} + \mathrm{Ric}_{22} - \mathrm{Ric}_{33} \\
&= R_{1212} + R_{1313} + R_{2121} + R_{2323} - R_{3131} - R_{3232} \\
&= 2R_{1212} = K(e_1, e_2)
\end{aligned}$$

Note that $\mathrm{Ric}_{ii} = \lambda\langle e_i, e_i\rangle = \lambda$, we have $K(e_1, e_2) = \lambda$ for any point p and any $e_1, e_2 \in T_pM$, with e_1, e_2 the normal orthogonal vectors at p.

So M has constant sectional curvature.

3. Let $f : M \to \mathbb{R}$ be a smooth function defined on a Riemannian manifold (M^n, g). Denote $\Sigma := f^{-1}(a)$ where a is a regular value of f. Show that the mean curvature H, with respect to the unit normal $N = -\frac{\nabla f}{|\nabla f|}$, of the hypersurface Σ is given by $H = \pm \mathrm{div} N$ (up to a sign depending on the sign convention in the definition of mean curvature). **Solution:**

Given $p \in \Sigma$, choose a orthonormal basis $\{e_1, \cdots, e_{n-1}\}$ of $T_p\Sigma$ at p. So the vectors $\{e_1, \cdots, e_{n-1}, N\}$ will form a orthonormal basis of T_pM. The mean

curvature H of Σ with respect to N is defined as

$$H = \sum_{i=1}^{n-1} \langle \nabla_{e_i} e_i, N \rangle$$

where we've extend $\{e_i\}$ to any local frame of Σ and $\nabla_X Y$ denote the coderivative on M. Since $\langle e_i, N \rangle \equiv 0$ on Σ, we have

$$H = \sum_{i=1}^{n-1} e_i \langle e_i, N \rangle - \sum_{i=1}^{n-1} \langle e_i, \nabla_{e_i} N \rangle = -\sum_{i=1}^{n-1} \langle e_i, \nabla_{e_i} N \rangle$$

Note that $\langle N, N \rangle = 1$ all the time. So

$$0 = N \langle N, N \rangle = 2 \langle N, \nabla_N N \rangle$$

Hence

$$H = -\sum_{i=1}^{n-1} \langle \nabla_{e_i} e_i, N \rangle - \langle \nabla_N N, N \rangle = -\mathrm{div} N = \mathrm{div} \frac{\nabla f}{|\nabla f|}.$$

(There might be a sign difference based on how to define the mean curvature and how to choose the normal.)

4. Consider the smooth map $F : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$F(u, v) = (\cos u, \sin u, \cos v, \sin v).$$

(a) Show that F is an isometric immersion (with respect to the flat metrics).

(b) Prove that the image of F lies inside the round 3-sphere $\mathbb{S}^3 := \{x \in \mathbb{R}^4 \mid |x|^2 = 2\}$, and $\Sigma = F(\mathbb{R}^2)$ is a minimal immersion into $\mathbb{S}^3$, equipped with the induced metric from $\mathbb{R}^4$.

Solution:

(a) We note the following identity

$$F_*(\frac{\partial}{\partial u}) = \frac{\partial F}{\partial u} = (-\sin u, \cos u, 0, 0),$$
$$F_*(\frac{\partial}{\partial v}) = \frac{\partial F}{\partial v} = (0, 0, -\sin v, \cos v)$$

So

$$\left\langle F_*(\frac{\partial}{\partial u}), F_*(\frac{\partial}{\partial u}) \right\rangle_{\mathbb{R}^4} = 1 = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle_{\mathbb{R}^2}$$
$$\left\langle F_*(\frac{\partial}{\partial u}), F_*(\frac{\partial}{\partial v}) \right\rangle_{\mathbb{R}^4} = 0 = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle_{\mathbb{R}^2}$$
$$\left\langle F_*(\frac{\partial}{\partial v}), F_*(\frac{\partial}{\partial v}) \right\rangle_{\mathbb{R}^4} = 1 = \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle_{\mathbb{R}^2}$$

Hence F is an isometric immersion.

(b) Note that $|F|^2 = \cos^2 u + \sin^2 u + \cos^2 v + \sin^2 v = 2$. So image of F lies inside $\mathbb{S}^3$.

Let N be a unit normal vector field along $F(\mathbb{R}^2)$ in $\mathbb{S}^3$. We use $\overline{\nabla}_X Y$ to denote the coderivative on $\mathbb{R}^4$ and $\nabla_X Y$ denote the coderivative on $\mathbb{S}^3$.

So the mean curvature can be calculate by following

$$\begin{aligned}
H &= \left\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, N \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v}, N \right\rangle \\
&= \left\langle \overline{\nabla}_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, N \right\rangle + \left\langle \overline{\nabla}_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v}, N \right\rangle \quad (\text{ since } N \in T\mathbb{S}^3) \\
&= \left\langle \frac{\partial^2 F}{\partial u^2}, N \right\rangle + \left\langle \frac{\partial^2 F}{\partial v^2}, N \right\rangle \\
&= \langle (-\cos u, -\sin u, 0, 0) + (0, 0, -\cos v, -\sin v), N \rangle = -\langle F(u, v), N \rangle \\
&= 0 \quad (\text{ since } N \perp F \in \mathbb{R}^4)
\end{aligned}$$

So Σ is a minimal immersion into $\mathbb{S}^3$.